The Best Invitation

Kerry M. Soileau

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Abstract

We consider the problem of being presented with up to n options in sequence, and determining when to stop and accept the latest option. Once rejected, an option will not again be offered. Each option has a known probability of being offered, but nothing is known about the sequence in which options will actually be offered. Each option has a known, finite positive value. We present an algorithm which maximizes the likelihood of accepting the best offered option.

Keywords. optimal, stopping, probability, algorithm.

1 Introduction

Suppose a young lady is eagerly anticipating the upcoming prom. Of course, she will need a date. For each of the n young men whom she thinks might ask her, she has estimated the probability that he will ask, together with his desirability, with the most desirable boy having desirability equal to 1, and every boy having a desirability greater than 0. These data are represented by the variables $p_1, p_2, p_3, \ldots, p_{n-1}, p_n$ and $x_1, x_2, x_3, \ldots, x_{n-1}$, respectively, with the indexing indicating desirability ordering, i.e. $i < j$ implies $x_i \leqslant x_j$. Since she has included only boys whom she thinks might ask her, all of the p_i

are positive. She doesn't know for certain which boys are willing to ask her. By "the boys who are willing to ask her," we mean the set of boys who would ask her if she rejected every boy who asked. She assumes that the boys who are willing to ask will do so in no particular order, that their invitations are independent of one another, and that each possible ordering of these boys' invitations is equally likely. The desirability of a boy who hasn't asked her is independent of the desirability of boys who have asked and been rejected. Once she refuses a boy, he will not ask again. If she happens to refuse every boy that asks, she will end up with no date for the prom, a payoff of 0. If $p_i < 1$ for every i, it's possible that no boy will ask, yielding her a payoff of 0. In this article we assume that her objective is to maximize the likelihood of accepting the invitation of the most desirable boy who is willing to ask her.

2 The First-ask-probability

In the following we will make use of a formula giving the probability that among the boys who have not asked, a particular boy will be the first among them to ask.

For probabilities and desirabilities $\{p_i\}_{i=1}^{n}$ and $\{x_i\}_{i=1}^{n-1}$,

$$P(k \text{ is the first boy to ask})$$

$$= p_k \left(1 - \frac{1}{2} \sum_{(j_1) \in M_{n,k,1}} p_{j_1} + \cdots + \frac{(-1)^{n-1}}{n} \sum_{(j_1, j_2, \ldots, j_{n-1}) \in M_{n,k,n-1}} p_{j_1} p_{j_2} \ldots p_{j_{n-1}} \right),$$

where

$$M_{n,k,r} = \{(j_1, j_2, \ldots, j_r); \{j_1, j_2, \ldots, j_r\} \subseteq \{1, 2, \ldots n\} \backslash \{k\}; j_1 < j_2 < \cdots < j_r\}.$$

For example, for $n = 2$,

$$P(1 \text{ is the first boy to ask}) = p_1 \left(1 - \frac{1}{2} \sum_{(j_1) \in \{(2)\}} p_{j_1} \right) = p_1 \left(1 - \frac{p_2}{2} \right)$$

and

$$P(2 \text{ is the first boy to ask}) = p_2 \left(1 - \frac{1}{2} p_1 \right) = p_2 \left(1 - \frac{p_1}{2} \right).$$

For $n = 3$,

$$P(1 \text{ is the first boy to ask})$$

$$= p_1 \left(1 - \frac{1}{2} \sum_{(j_1) \in \{(j_1); \{j_1\} \subseteq \{2,3\}\}} p_{j_1} + \frac{1}{3} \sum_{(j_1, j_2) \in \{(2,3)\}} p_{j_1} p_{j_2} \right),$$

$$= p_1 \left(1 - \frac{1}{2} (p_2 + p_3) + \frac{1}{3} p_2 p_3 \right),$$

For convenience, in the following, we write

$$f_{n,k}(p_1, p_2, \ldots p_n) \equiv P(k \text{ is the first boy to ask})$$

$$= p_k \left(1 - \frac{1}{2} \sum_{(j_1) \in M_{n,k,1}} p_{j_1} + \cdots + \frac{(-1)^{n-1}}{n} \sum_{(j_1, j_2, \ldots, j_{n-1}) \in M_{n,k,n-1}} p_{j_1} p_{j_2} \ldots p_{j_{n-1}} \right),$$

3 Approach

We propose an algorithm for deciding whether to accept or reject the invitation of a given boy. We regard the girl's problem as playing a sequence of games. The first game presents a boy who has asked her, and $n-1$ who have not. She has two available plays in this game: accept or reject his invitation. If she accepts, the game is over and her payoff is equal to the desirability of the accepted boy. If she rejects him, she is presented with a new game. This game now consists of one of the remaining boys as the asker, and the $n-2$ boys who haven't asked. We assume that the desirability values for each of the $n-1$ boys are unchanged. However, as we will see below, this is generally not the case for the ask-probability values. She continues playing this sequence of games, each with fewer boys than the previous one, doing the appropriate Bayesian update on the probabilities when passing from one game to the next. This continues until either she accepts some boy or no boy remains who is willing to ask her.

Explicitly, the game begins, for the case of three boys, with the following data:

Boy	x_i	p_i
1	x_1	p_1
2	x_2	p_2
3	1	p_3

If boy 1 asks first and she rejects him, the game becomes

Boy	x_i	p_i
2	x_2	$\frac{p_2(3-p_3)}{p_2(2p_3-3)-3p_3+6}$
3	1	$\frac{(3-p_2)p_3}{p_2(2p_3-3)-3p_3+6}$

If boy 2 asks first and she rejects him, the game becomes

Boy	x_i	p_i
1	x_1	$\frac{p_1(3-p_3)}{p_1(2p_3-3)-3p_3+6}$
3	1	$\frac{(3-p_1)p_3}{p_1(2p_3-3)-3p_3+6}$

If boy 3 asks first and she rejects him, the game becomes

Boy	x_i	p_i
1	x_1	$\frac{p_1(3-p_2)}{p_1(2p_2-3)-3p_2+6}$
2	x_2	$\frac{(3-p_1)p_2}{p_1(2p_2-3)-3p_2+6}$

To see why each boy's ask-probability will, in general, vary from game to game in the sequence, we give a simple example. Assuming the ask-probability of each of the boys is $\frac{1}{2}$, the relative probabilities of any possible series of invitations are represented by the entries in the table below. That is, the probability of a series of invitations is proportional to the number of its appearances in the table. $\emptyset$ is the case in which no boys ask, 12 means boy 1 asks first, then boy 2, but boy 3 doesn't ask, and so on.

∅	1	2	3	12	21	31	123
∅	1	2	3	12	21	31	132
∅	1	2	3	12	21	31	213
∅	1	2	3	13	23	32	231
∅	1	2	3	13	23	32	312
∅	1	2	3	13	23	32	321

Now suppose, for instance, that boy 2 asks first. The table

		2			21		
		2			21		
		2			21		213
		2			23		231
		2			23		
		2			23		

represents the possibilities, given that boy 2 asked first, for the set of boys who are willing to ask and in what order they would ask. The three entries of 21 and the entries 213 and 231 represent the outcome that boy 1 would eventually ask, for a total of 5 entries. Dividing that by the total of 14 equally likely outcomes yields a conditional probability of $\frac{5}{14}$. Thus boy 2 having asked first has slightly altered the probability that boy 1 would eventually ask.

3.1 The Algorithm

Each time a boy, say boy k, asks, she accepts boy k if the probability that no more desirable boy will ask her is greater than or equal to $\frac{1}{2}$, else she rejects boy k and the game continues. To put it another way, she should accept the first boy for which it is not more likely than not that some better boy will ask her.

Explicitly, she accepts boy k if and only if

$$P\left(\text{boys } k+1, k+2, \ldots, n \text{ will not ask}|\text{boy } k \text{ asks first}\right) \geqslant \frac{1}{2}$$

If she rejects boy k, the remaining $n-1$ boys form, after the appropriate reindexing of the desirabilities and Bayesian update of the probabilities, a new problem of the same type, and the first of these boys to ask can be accepted or refused according to a reapplication of the above criterion. This iteration continues until some boy is accepted via the criterion, or all boys who are willing to ask are refused.

This can be illustrated by the following simple example. Let $n = 2$, with data $\{p_1, p_2\}$ and $\{x_1, 1\}$. Suppose boy 1 has asked.

$$\begin{aligned}&P(\text{boy 2 will not ask}|\text{boy 1 asked first})\\ &= \frac{P(\text{boy 2 will not ask and boy 1 asked first})}{P(\text{boy 1 asked first})} = \frac{p_1(1-p_2)}{p_1 - 1/2p_1p_2} = \frac{2-2p_2}{2-p_2}\end{aligned}$$

According to the criterion, she should accept boy 1 if and only if

$$P(\text{boy 2 will not ask}|\text{boy 1 asked first}) \geqslant \frac{1}{2}$$

i.e.

$$\frac{2-2p_2}{2-p_2} \geqslant \frac{1}{2},$$

which is equivalent to

$$p_2 \leqslant \frac{2}{3}.$$

3.2 Bayesian Update

As mentioned before, the first boy that asks her will alter the probabilities that the other boys will ask her eventually.

Proposition 3.1 *The updated probability that boy j would eventually ask, given that boy j has not yet asked and boy k asked first, is given by*

$$p_{j|k} = \frac{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n\right)}{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n\right)} p_j.$$

Proof

$$\begin{aligned}
&P(\text{boy } j \text{ would eventually ask}|\text{boy } k \text{ asked first}) \\
&= \frac{P(\text{boy } j \text{ would eventually ask and boy } k \text{ asked first})}{P(\text{boy } k \text{ asked first})} \\
&= \frac{P(\text{boy } k \text{ asked first and boy } j \text{ would eventually ask})}{P(\text{boy } k \text{ asked first})}
\end{aligned}$$

$$= \frac{P(\text{boy } k \text{ asked first}|\text{boy } j \text{ would eventually ask})P(\text{boy } j \text{ would eventually ask})}{P(\text{boy } k \text{ asked first})}.$$

Since $P(\text{boy } k \text{ asked first}|\text{boy } j \text{ would eventually ask})$

$$= f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n\right),$$

$$P(\text{boy } j \text{ would eventually ask}) = p_j,$$

and $P(\text{boy } k \text{ asked first}) = f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n\right)$, we get that

$$P(\text{boy } j \text{ would eventually ask}|\text{boy } k \text{ asked first})$$

$$= \frac{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n\right)}{f_{n,k}\left(p_1, p_2, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n\right)} p_j.$$

3.3 The Acceptance Criterion

As we stated earlier, she should accept boy k if and only if

$$P\left(\text{boys } k+1, k+2, \ldots, n \text{ will not ask}|\text{boy } k \text{ asks first}\right) \geqslant \frac{1}{2}.$$

Note that

$$P\left(\text{boys } k+1, k+2, \ldots, n \text{ will not ask}|\text{boy } k \text{ asks first}\right)$$

$$= \frac{P\left(\text{boys } k+1, k+2, \ldots, n \text{ will not ask, and boy } k \text{ asks first}\right)}{P\left(\text{boy } k \text{ asks first}\right)}$$

$$= \left(\frac{P\left(\text{boy } k \text{ asks first}|\text{boys } k+1, k+2, \ldots, n \text{ will not ask}\right)}{P\left(\text{boy } k \text{ asks first}\right)}\right)$$

$$\cdot P\left(\text{boys } k+1, k+2, \ldots, n \text{ will not ask}\right)$$

$$= \frac{P\left(\text{boy } k \text{ asks first} | \text{boys } k+1, k+2, \ldots, n \text{ will not ask}\right)}{P\left(\text{boy } k \text{ asks first}\right)}(1-p_{k+1})(1-p_{k+2}) \ldots (1-p_n)$$

$$= \frac{f_{n,k}\left(p_1, p_2, \ldots, p_k, 0, \ldots, 0\right)}{f_{n,k}\left(p_1, p_2, \ldots, p_k, p_{k+1}, \ldots, p_n\right)}(1-p_{k+1})(1-p_{k+2}) \ldots (1-p_n)$$

4 A Detailed Example

We conclude with an example. Let the boys' data be as in the following table:

Boy	x_i	p_i
Al	.6600	.8300
Ben	.8700	.2700
Carl	.8800	.5600
Don	1.0000	.8500

Suppose the first boy to ask her (in this fictitious scenario) is Carl. Having observed Carl asking first, she does a Bayesian update of the probabilities for Al, Ben and Don later asking and obtains

Boy	x_i	p_i'
Al	.6600	.7586
Ben	.8700	.2069
Don	1.0000	.7836

Now we consider the probability that some boy more desirable than Carl will ask. Note that Carl's desirability is .8800, so only Don is more desirable.

The probability that Carl is the best boy who will ask, i.e. that Don will not ask is $0.216374 < \frac{1}{2}$, thus it is more likely than not that Carl is not the best boy who will ask her, and thus he is rejected.

Now suppose Al asks next. Using the above data, having observed that Al has asked, she does a Bayesian update of the probabilities for Ben and Don later asking and obtains

Boy	x_i	p''_i
Ben	.8700	.1368
Don	1.0000	.6528

Now we consider the probability that some boy more desirable than Al will ask. Note that Al's desirability is .6600, so only Ben and Don are more desirable. The probability that Al is the best boy who will ask, i.e. that neither Ben nor Don will ask, is $0.307098 < \frac{1}{2}$, thus it is more likely than not that Al is not the best boy who will ask her, so he is rejected.

Suppose Don asks next. Since Ben's desirability is less that Don's, the probability that Don is the best boy who will ask is $1.000000 \geqslant \frac{1}{2}$, thus he is accepted and the game ends.

5 Bibliography

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